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WHAT IS THE MOST EFFICIENT HIGH-ENERGY ACCELERATOR?*

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ABSTRACT

The accelerator configuration that will result in the largest fraction of accelerator kinetic energy transferred to accelerated particles is explicitly determined from general principles.

DISCUSSION

The question posed in the title of this paper can be interpreted in various ways. Rather than survey all the types of accelerators that exist or have been proposed in search of a candidate for the most efficient accelerator, I will propose a specific definition for "most efficient accelerator" and then use this definition to determine what the most efficient accelerator should look like. Surprisingly, if we consider only the kinetic energy in the accelerator, a definite answer emerges. An answer to this question that includes the contribution of field energy will be considerably more difficult to find and is certainly fraught with difficulties, both in determining what constitutes an acceptable accelerator and with certain mathematical problems of a technical nature.

By "most efficient accelerator" I mean an accelerator such that the greatest fraction of accelerator energy is extracted by the passage of a single bunch of particles. This definition obviously begs the question of whether or not higher efficiency might not be obtained by energy storage and extraction over the duration of many bunches. Though I do not address the question of multiple bunches and efficiency, it may be amenable to the kind of analysis I present here.

I also define "high-energy" to mean that the relativistic factor, γ, of the particles to be accelerated is already large enough that it changes by only a small amount as it passes a given point in the accelerator. Precisely how large this is will be specified presently.

Accelerator energy to be extracted can be categorized as either field energy, i.e., total $(E^2 + B^2)/8\pi$, or as kinetic energy of particles composing the accelerator. The energy in most accelerators is a mixture of both field energy and kinetic energy, i.e., the field energy in a cavity and the kinetic energy of the particles comprising the currents in the walls. To analyze the contribution of field energy is technically difficult because of various spurious singularities that can arise; for example, by placing two charged particles arbitrarily close together, one can, classically, extract arbitrarily large amounts of energy with arbitrarily small displacements. Whether one should regularize these singularities with a quantum model or by smoothing as in a

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plasma model or in some other fashion is not clear and may not, in fact, have a single answer independent of circumstances. Because of these ambiguities of formulation and difficulties in technical detail, I will ignore field energy and treat only the kinetic energy.

The question I have posed can now, within the context of the above assumptions, be rephrased as "What distribution of kinetic energy among accelerator particles will result in the maximum fraction of that kinetic energy transferred to accelerated particles?" In other words, we only have to analyze the interaction of a single accelerator particle with the accelerated bunch to determine what initial velocity at that initial position will result in the maximum transfer of energy to the bunch. It is obvious that the answer is nonzero because a particle at rest can only gain energy from the bunch, not impart energy to it. That we can ignore the influence on the trajectory of an accelerator particle of the other accelerator particles is partially justified by our neglect of field energy and will be discussed after the calculations.

DERIVATION

Though the question, now, is perfectly well specified and could, in principle, be answered in general, I will make a number of additional nonessential assumptions to simplify the analysis and arrive at an explicit answer. These assumptions are (1) the accelerating gradient is small enough that the bunch particles can be assumed to follow straight-line constant-energy trajectories during the interaction (this is really a limitation on the total number of particles supplying the energy), (2) the particles supplying the energy can be treated nonrelativistically (this turns out to involve a limitation on the total number of particles in the bunch), (3) the accelerator particles are arranged about the bunch in an azimuthally symmetric configuration, and (4) the accelerated bunch has transverse dimensions small enough to be ignored. Having made these assumptions, I can now investigate the motion of an accelerator particle under the influence of the fields of the accelerated bunch that can be assumed to have a straight-line constant-energy trajectory. The equation of motion of such a particle is

$$m \frac{d\vec{v}}{dt} = e\vec{E} \quad , \tag{1}$$

where m is the mass, e is the charge, $\vec{E}$ is the electric field of the bunch, and where we have ignored the force from the magnetic field because of the assumption of nonrelativistic motion and because the electric and magnetic fields of the bunch will be about equal in magnitude. The total energy transfer to this particle from the bunch is

$$\Delta\varepsilon = e\int_{-\infty}^{\infty} dt \; \vec{v}(t) \cdot \vec{E}(t) \quad , \tag{2}$$

where $\vec{v}(t)$ is the solution of Eq. (1) and $\vec{E}(t)$ is the electric field of the bunch evaluated at the position of the particle at time t. The solution to Eq. (1) can be written as

$$\vec{v}(t) = \vec{v}_0 + \int_{-\infty}^{t} dt' \, \frac{eE(t')}{m} \quad . \tag{3}$$

If we let the z-axis be the path of the accelerated bunch and assume that the accelerator particle lies in the x-z plane, then by using Eq. (3) in Eq. (2), we find

$$\Delta\varepsilon = e\int_{-\infty}^{\infty} dt \, [E_x(t)v_{ox} + E_z(t)v_{oz}]$$

$$+ \frac{e^2}{m}\int_{-\infty}^{\infty} dt[E_x(t)\int_{-\infty}^{t} dt'E_x(t') + E_z(t)\int_{-\infty}^{t} dtE_z(t')] \quad . \tag{4}$$

If we assume that the center of the bunch passes z = 0 at t = 0, then the bunch can be described in terms of a distribution, $g(\tau)$, where g is normalized to 1 and is the relative density of particles that cross z = 0 at t = τ. The electric fields due to the bunch are then the integrals over the bunch of the well-known[1] fields due to a single relativistic particle; i.e.,

$$E_x(t) = Ne\gamma \int d\tau \, \frac{g(\tau) \; x(t)}{[x^2(t) + \gamma^2 v^2(t + \tau - \frac{z(t)}{v})^2]^{3/2}} \tag{5}$$

and

$$E_z(t) = Nev \int d\tau \, \frac{g(\tau) \; (t + \tau - \frac{z(t)}{v})}{[x^2(t) + \gamma^2 v^2(t + \tau - \frac{z(t)}{v})^2]^{3/2}} \quad , \tag{6}$$

where N is the total number of particles in the bunch, and x(t) and z(t) are the coordinates of the accelerator particle. Thus, we now need x(t) and z(t), which, through Eq. (3), depend on E(t). To solve these equations, we make an expansion and use our assumption that there are not too many particles in the bunch. Therefore, the electric fields are not large enough to cause a change in transverse position comparable to the transverse coordinate, x_o, that the accelerator particle would have at t = 0 in the absence of the bunch. The zero-order terms, i.e., the result assuming stationary accelerator particles, in this expansion contribute the main effect, but I keep through the first-order terms to confirm that the corrections are indeed small.

Before I begin the expansion, I will collect all dimensional quantities into coefficients by defining dimensionless variables as follows:

$$\tilde{\tau} \equiv \frac{\gamma v \tau}{x_o} \quad , \tag{7}$$

$$\bar{g} \equiv \frac{g x_o}{\gamma v} \, , \tag{8}$$

$$\bar{t} \equiv \frac{\gamma v t}{x_o} \, , \tag{9}$$

$$\bar{x} \equiv \frac{x(t)}{x_o} - 1 \quad , \quad \text{and} \tag{10}$$

$$\bar{z} \equiv \frac{\gamma z(t)}{x_o} \, , \tag{11}$$

where, with the above assumptions, $\bar{x}$ and $\bar{z}$ are small expansion quantities. With these definitions, the electric fields, Eqs. (5) and (6), can be expressed as

$$E_x(\bar{t}) = \frac{Ne\gamma(1 + \bar{x})}{x_o^{\,2}} \int d\bar{\tau} \, \frac{\bar{q}(\bar{\tau})}{[(1 + \bar{x})^2 + (\bar{t} + \bar{\tau} - \bar{z})^2]^{3/2}} \, , \tag{12}$$

and

$$E_z(\bar{t}) = \frac{-Ne}{x_o^{\,2}} \int d\bar{\tau} \, \frac{\bar{q}(\bar{\tau})\,(\bar{t} + \bar{\tau} - \bar{z})}{[(1 + \bar{x})^2 + (\bar{t} + \bar{\tau} - \bar{z})^2]^{3/2}} \, , \tag{13}$$

where the integrals are now dimensionless and are of order unity.

Let us begin the expansion about ($\bar{x} = 0$, $\bar{z} = 0$). To zeroth and first order, Eqs. (12) and (13) become

$$E_x(\bar{t}) \simeq \frac{Ne\gamma}{x_o^{\,2}} \left\{ (1 + \bar{x}) \int d\bar{\tau} \, \frac{\bar{q}(\bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{3/2}} \right.$$
$$\left. -3\bar{x} \int d\bar{\tau} \, \frac{q(\bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} + 3\bar{z} \int d\bar{\tau} \, \frac{\bar{q}(\bar{\tau})\,(\bar{t} + \bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} \right\} \, , \tag{14}$$

and

$$E_z(\bar{t}) \simeq \frac{Ne\gamma}{x_o^{\,2}} \left\{ -\int d\bar{\tau} \, \frac{\bar{q}(\bar{\tau})\,(\bar{t} + \bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{3/2}} \right.$$
$$+ 3\bar{x} \int d\bar{\tau} \, \frac{\bar{q}(\tau)\,(\bar{t} + \bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} + \bar{z} \int d\bar{\tau} \, \frac{\bar{q}(\bar{\tau})}{[1 + (\bar{t} + \bar{\tau})\,]^{2\;3/2}}$$
$$\left. - 3\bar{z} \int d\ddot{\tau} \, \frac{\bar{q}(\bar{\tau})\,(\bar{t} + \bar{\tau})^2}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} \right\} \, . \tag{15}$$

Using Eq. (3) and the definitions of dimensionless quantities, it is easy to see that, to first order, the coordinates $\bar{x}$ and $\bar{z}$ can be expressed as

$$\bar{x} \simeq \frac{v_{ox}\bar{t}}{\gamma v} + \frac{Ne^2}{\gamma m v^2 x_o}\int_{-\infty}^{\bar{t}} d\bar{t}' \int_{-\infty}^{\bar{t}'} d\bar{t}'' \int d\bar{\tau}\, \frac{\bar{q}(\bar{\tau})}{[1 + (t'' + \bar{\tau})^2]^{3/2}} \quad , \tag{16}$$

and

$$\bar{z} \simeq \frac{v_{oz}\bar{t}}{\gamma v} - \frac{Ne^2}{\gamma m v^2 x_o}\int_{-\infty}^{\bar{t}} d\bar{t}' \int_{-\infty}^{\bar{t}'} d\bar{t}'' \int d\bar{\tau}\, \frac{\bar{q}(\bar{\tau})\,(\bar{t}'' + \bar{\tau})}{[1 + (\bar{t}'' + \bar{\tau})^2]^{3/2}} \quad . \tag{17}$$

Note that the electric fields have zero-order terms that survive even when $\bar{x}$, $\bar{z} \to 0$. Note also that $\bar{x}$ and $\bar{z}$ are proportional to $1/\gamma$ and thus vanish at very high bunch energies.

To facilitate further manipulations, let us define the following functions:

$$h_1(\bar{t}) \equiv \int d\bar{\tau}\, \frac{\bar{q}(\bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{3/2}} \quad , \tag{18}$$

$$h_2(\bar{t}) \equiv \int d\bar{\tau}\, \frac{\bar{q}(\bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} \quad , \tag{19}$$

$$h_3(\bar{t}) \equiv \int d\bar{\tau}\, \frac{q(\bar{\tau})\,(\bar{t} + \bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} \quad , \tag{20}$$

$$h_4(\bar{t}) \equiv \int d\bar{\tau}\, \frac{q(\bar{\tau})\,(\bar{t} + \bar{\tau})}{[1 + (\bar{t} + \bar{\tau})^2]^{3/2}} \quad , \tag{21}$$

and

$$h_5(\bar{t}) \equiv \int d\bar{\tau}\, \frac{\bar{q}(\bar{\tau})\,(\bar{t} + \bar{\tau})^2}{[1 + (\bar{t} + \bar{\tau})^2]^{5/2}} \quad . \tag{22}$$

With these definitions we find that Eqs. (14), (15), (16), and (17) can be rewritten as

$$E_x(\bar{t}) \simeq \frac{Ne\gamma}{x_o^2}\,[(1 + \ddot{x})h_1 - 3\bar{x}h_2 + 3\bar{z}h_3] \quad , \tag{23}$$

$$E_z(\bar{t}) \simeq \frac{Ne}{x_o^2}\,[-h_4 + 3\bar{x}h_3 + \bar{z}h_1 - 3\bar{z}h_5] \quad , \tag{24}$$

$$\bar{x} \simeq \frac{v_{ox}\bar{t}}{\gamma v} + \frac{Ne^2}{\gamma m v^2 x_o} \int_{-\infty}^{\bar{t}} d\bar{t}' \int_{-\infty}^{\bar{t}'} d\bar{t}'' \; h_1(\bar{t}'') \quad , \tag{25}$$

and

$$\bar{z} \simeq \frac{v_{oz}\bar{t}}{v} - \frac{Ne^2}{\gamma m v^2 x_o} \int_{-\infty}^{\bar{t}} d\bar{t}' \int_{-\infty}^{\bar{t}'} d\bar{t}'' \; h_4(\bar{t}'') \quad . \tag{26}$$

Using Eqs. (25) and (26) in Eqs. (23) and (24) and putting the results in Eq. (4) we find that the energy change can be written as

$$\Delta\varepsilon \simeq \frac{Ne^2}{x_o}\left[\frac{v_{ox}\alpha_1}{v} + \frac{v_{oz}\alpha_2}{\gamma v} + \frac{Ne^2}{m v^2 x_o}\left(\alpha_3 + \frac{\alpha_4}{\gamma^2}\right)\right] \quad , \tag{27}$$

where we have defined

$$\alpha_1 \equiv \int_{-\infty}^{\infty} d\bar{t}[(1+\bar{x})h_1 - 3\bar{x}h_2 + 3\bar{z}h_3] \quad , \tag{28}$$

$$\alpha_2 \equiv \int_{-\infty}^{\infty} d\bar{t}(-h_4 + 3\bar{x}h_3 + \bar{z}h_1 - 3\bar{z}h_5) \quad , \tag{29}$$

$$\alpha_3 \equiv \int_{-\infty}^{\infty} d\bar{t}[(1+\bar{x}) - 3\bar{x}h_2 + 3\bar{z}h_3]\int_{-\infty}^{\bar{t}} d\bar{t}'[(1+\bar{x}) - 3\bar{x}h_2 + 3\bar{z}h_3] \quad , \tag{30}$$

and

$$\alpha_4 \equiv \int_{-\infty}^{\infty} d\bar{t}(-h_4 + 3\bar{x}h_3 + \bar{z}h_1 - 3\bar{z}h_5)$$

$$\times \int_{-\infty}^{\bar{t}} d\bar{t}'(-h_4 + 3\bar{x}h_3 + \bar{z}h_1 - 3\bar{z}h_5) \quad . \tag{31}$$

Note that these coefficients contain zeroth-, first-, and second-order terms. To be consistent, we should drop the quadratic terms; let us define

$$\alpha_{10} \equiv \int_{-\infty}^{\infty} h_1 d\bar{t} \quad , \tag{32}$$

$$\alpha_{20} \equiv -\int_{-\infty}^{\infty} h_4 d\bar{t} \quad , \tag{33}$$

$$\alpha_{30} \equiv \int_{-\infty}^{\infty} h_1 d\bar{t} \int_{-\infty}^{\bar{t}} h_1 d\bar{t}' \quad , \tag{34}$$

$$\alpha_{40} \equiv \int_{-\infty}^{\infty} h_4 d\bar{t} \int_{-\infty}^{\bar{t}} h_4 d\bar{t}' \quad , \tag{35}$$

$$\alpha_{11} \equiv \int_{-\infty}^{\infty} d\bar{t}[\bar{x}(h_1 - 3h_2) + 3\bar{z}h_3] \quad , \tag{36}$$

$$\alpha_{21} \equiv \int_{-\infty}^{\infty} d\bar{t}[3\bar{x}h_3 + \bar{z}(h_1 - 3h_5)] \quad , \tag{37}$$

$$\alpha_{31} \equiv \int_{-\infty}^{\infty} d\bar{t}\{[\bar{x}(h_1 - 3h_2) + 3\bar{z}h_3]\int_{-\infty}^{\bar{t}} d\bar{t}' h_1$$

$$+ h_1 \int_{-\infty}^{\bar{t}} d\bar{t}'[\bar{x}(h_1 - 3h_2) + 3\bar{z}h_3]\} \quad , \tag{38}$$

and

$$\alpha_{41} \equiv -\int_{-\infty}^{\infty} d\bar{t}\{[3\bar{x}h_3 + \bar{z}(h_1 - 3h_5)]\int_{-\infty}^{\bar{t}} d\bar{t}' h_4$$

$$+ h_4 \int_{-\infty}^{\bar{t}} d\bar{t}'[3\bar{x}h_3 + \bar{z}(h_1 - 3h_5)]\} \quad , \tag{39}$$

so that in this approximation

$$\alpha_1 \simeq \alpha_{10} + \alpha_{11} \quad , \tag{40}$$

$$\alpha_2 \simeq \alpha_{20} + \alpha_{21} \quad , \tag{41}$$

$$\alpha_3 \simeq \alpha_{30} + \alpha_{31} \quad , \tag{42}$$

and

$$\alpha_4 \simeq \alpha_{40} + \alpha_{41} \quad . \tag{43}$$

We now want to maximize the function $f \equiv -\Delta\varepsilon/\varepsilon_0$ (ε_0 is the initial energy of an accelerator particle, $\varepsilon_0 = mv_0^2/2$) with respect to the initial velocity, v_o; this will maximize the fraction of energy transferred to the accelerated bunch. Continuing with the expansion, we write

$$\Delta\varepsilon \approx \Delta\varepsilon_o + \Delta\varepsilon_1 \quad , \tag{44}$$

and

$$\vec{v}_o \approx \vec{w}_o + \vec{w}_1 \quad , \tag{45}$$

where the subscripts on the right-hand sides refer to the order of the term. To maximize f we require that

$$\frac{\partial f}{\partial \vec{v}_o} = 0 \quad . \tag{46}$$

It is easy to confirm that to lowest order, Eq. (46) becomes

$$\frac{\partial \Delta\varepsilon_o}{\partial \vec{v}_o} = \frac{2\Delta\varepsilon_o w_o}{w_o^2} \quad . \tag{47}$$

Using Eqs. (27) and (40)-(43) in Eq. (47), a little calculation reveals that the maximum condition becomes

$$\alpha_{10} = \frac{2w_{ox}}{w_o^2}\left[w_{ox}\,\alpha_{10} + \frac{w_{oz}\alpha_{20}}{\gamma} + \frac{Ne^2\alpha_{30}}{mvx_o} + \frac{Ne^2\alpha_{40}}{\gamma mvx_o}\right] \quad , \tag{48}$$

and

$$\frac{\alpha_{20}}{\gamma} = \frac{2w_{oz}}{w_o^2}\left[w_{ox}\,\alpha_{10} + \frac{w_{oz}\alpha_{20}}{\gamma} + \frac{Ne^2\alpha_{30}}{mvx_o} + \frac{Ne^2\alpha_{40}}{\gamma mvx_o}\right] \quad . \tag{49}$$

Taking the ratio of Eqs. (49) and (48) gives

$$\frac{w_{oz}}{w_{ox}} = \frac{\alpha_{20}}{\gamma\alpha_{10}} \quad . \tag{50}$$

Thus, we see that the z component of the velocity is smaller than the radial component by at least $1/\gamma$. In addition, if we assume for simplicity that the bunch is symmetric about its center, i.e.,

$$\bar{g}(-\bar{\tau}) = \bar{g}(\bar{\tau}) \quad , \tag{51}$$

then it is easy to see that $\alpha_{20} = 0$ and, thus, that $w_{oz} = 0$. Therefore, with the assumption of symmetry, Eq. (48) has the solution

$$w_{ox} = \frac{-2Ne^2\alpha_{30}}{mvx_o\alpha_{10}} \quad , \tag{52}$$

where we have used the fact that α_{40} also vanishes under the assumption of symmetry. Inserting these solutions in the expression for the fractional energy extracted, we find that to lowest order, the fraction is

$$f_o = \frac{\alpha_{10}^2}{2\alpha_{30}} \quad . \tag{53}$$

Finally, using the assumption of symmetry, Eq. (51), one can easily show that

$$\alpha_{30} = \frac{\alpha_{10}^2}{2} \quad , \tag{54}$$

so that the maximum extracted fraction is $f_o = 1$ (i.e., 100%). The initial velocity required to produce this optimum efficiency can be obtained from Eqs. (52) and (54) and is given by

$$w_{oz} = 0 \quad , \text{ and} \tag{55}$$

$$w_{ox} = \frac{-Ne^2\alpha_{10}}{mvx_o} \quad , \tag{56}$$

where α_{10} is given by Eq. (32). In summary, the optimum accelerator configuration is one in which the accelerator particles are moving purely radially (for a symmetric bunch) with a velocity proportional to the number of particles in the accelerated bunch and inversely proportional to the distance of the accelerator particle from the bunch axis. As a numerical example, let us consider the following case:

$$N = 10^{10} \quad , \tag{57}$$

$$\alpha_{10} \simeq 1 \quad , \tag{58}$$

$$v = c \quad , \text{ and} \tag{59}$$

$$x_o = 1 \text{ mm} \quad . \tag{60}$$

We then find that

$$\frac{w_{ox}}{c} = 0.028 \quad , \tag{61}$$

thus confirming, a posteriori, the nonrelativistic assumption.

Carrying out this expansion to any order is completely straightforward but tedious. For completeness, we can give the first-order correction to w_o, i.e., w_1.

$$w_{1x} = \frac{2Ne^2}{mvx_o}\left[\alpha_{11} - \frac{\alpha_{31}}{\alpha_{10}} - \frac{\alpha_{41}}{\gamma^2\alpha_{10}} - \frac{Ne^2}{mvx_o}\left(2\alpha_{10}\frac{\partial\alpha_{11}}{\partial v_{ox}} + \frac{\partial\alpha_{31}}{\partial v_{ox}} + \frac{1}{\gamma^2}\frac{\partial\alpha_{41}}{\partial v_{ox}}\right)\right] \quad , \tag{62}$$

and

$$w_{1z} = \frac{-2N^2e^4}{m^2v^2x_o^2}\left(-2\alpha_{10}\frac{\partial\alpha_{11}}{\partial v_{oz}} + \frac{\partial\alpha_{31}}{\partial v_{oz}} + \frac{1}{\gamma^2}\frac{\partial\alpha_{41}}{\partial v_{oz}}\right) \quad . \tag{63}$$

If in Eqs. (62) and (63) we insert the expressions (36)-(39) for α_{i1} and expressions (25) and (26) for x and z, we see that the ratio of w_1 to w_o is approximately

$$\left|\frac{w_1}{w_o}\right| \simeq \frac{Ne^2}{\gamma mv^2x_o} \tag{64}$$

and thus is small when the right hand side is small. As an example, taking our previous parameters (57)-(60), we find that for the right-hand side of Eq. (64) to be small, we must require

$$\gamma > 0.028 \quad , \tag{65}$$

which is obviously true. We thus see that the nonrelativistic restriction, i.e., Eq. (56) is small, is more stringent than the condition that the lowest-order term dominate.

To evaluate the condition that the change in bunch energy be small compared to the energy, we first observe that the time T for a bunch particle to pass an accelerator particle is approximately

$$T \simeq \frac{x_o}{\gamma c} \quad . \tag{66}$$

The amount of energy given up by an accelerator particle at x_o is

$$\Delta\epsilon \simeq \frac{N^2e^4}{2mv^2x_o^2} \quad , \tag{67}$$

so that if the number density of accelerator particles at that x_o is $n(x_o)$, the total energy gained by the bunch from particles at that location is

$$\Delta\varepsilon_{bunch} \simeq 2\pi \int cTx_o n(x_o) \frac{N^2e^4}{2mv^2x_o^2} dx_o \quad . \tag{68}$$

Thus, the energy gain per bunch particle compared to its energy is

$$\frac{\Delta\gamma}{\gamma} \simeq \frac{\pi Ne^4}{\gamma^2m^2v^4} \int n(x_o)dx_o \quad , \tag{69}$$

which should be small for our approximations to be justified.

If an accelerator particle is acted on by potentials from other accelerator components then the amount of energy transfer to the bunch will be essentially unchanged if this energy transfer is large compared to the difference in potential in the distance the particle moves in time T [Eq. (66)]. Thus, our analysis will have greater validity than the initial assumptions seemed to imply under most circumstances, though extreme accelerator conditions with steep gradients in potential may require further analysis.

In summary, I have found, considering only the kinetic energy, the optimum accelerator configuration and the maximum efficiency that can be obtained. Clearly a number of refinements to my calculation could be made; in particular, the effects of bunch asymmetry and transverse bunch size could be easily taken into account. Whether such a general analysis can be done including the field energy is still open to question. In any event, it is tempting to conjecture that no other configuration with an efficiency this high (100%) exists. Whether the configuration I have found can be realized or is practical is a question for future study.

REFERENCE

1. J.D. Jackson, Classical Electrodynamics, (John Wiley & Sons, New York, 1962), p. 381.